

MG13 proceedings: Construction of gauge-invariant variables for linear-order metric perturbations on an arbitrary background spacetime

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An outline of a proof of the decomposition of the linear metric perturbation into gauge-invariant and gauge-variant parts on an arbitrary background spacetime is discussed through an explicit construction of gauge-invariant and gauge-variant parts. Although this outline is incomplete, yet, due to our assumptions, we propose a conjecture which states that the linear metric perturbation is always decomposed into its gauge-invariant and gauge-variant parts. If this conjecture is true, we can develop the higher-order gauge-invariant perturbation theory on an arbitrary background spacetime.

Keywords: higher-order perturbations, gauge-invariance, arbitrary background spacetime

1. Introduction

As well-known, general relativity is based on general covariance and the “gauge degree of freedom”, which is unphysical degree of freedom of perturbations, arises due to this general covariance. Furthermore, gauge-transformation rules for higher-order perturbations are very complicated. So, it is worthwhile to investigate higher-order gauge-invariant perturbation theory from a general point of view.

According to this motivations, we have been formulating the higher-order general-relativistic gauge-invariant perturbation theory.¹ These works are based on the single assumption that *we already know the procedure to find gauge-invariant variables for linear-order metric perturbations*. (Conjecture 2.1 in this article) and our formulation is well-defined except for this assumption.

The main purpose of this article is to give a brief outline of a proof of this assumption.²

2. Perturbations in general relativity and gauge-invariant variables

Here, we concentrate on the second-kind gauge in perturbation theories with general covariance.¹ In perturbation theories, we always treat two spacetime manifolds. One is the physical spacetime \mathcal{M}_λ which is our nature itself and another is the background spacetime \mathcal{M}_0 which is prepared by hand for perturbative analyses. *The gauge choice of the second kind* is the point identification map $\mathcal{X}_\lambda : \mathcal{M}_0 \mapsto \mathcal{M}_\lambda$. *The gauge transformation of the second kind* is a change $\mathcal{X}_\lambda \rightarrow \mathcal{Y}_\lambda$ of this identification.

Once we specify a gauge choice \mathcal{X}_λ , we can define perturbations of a physical variable \bar{Q}_λ using the pulled-back $\mathcal{X}_\lambda^* \bar{Q}$ of \bar{Q}_λ . $\mathcal{X}_\lambda^* \bar{Q}$ is expanded as

$$\mathcal{X}_\lambda^* \bar{Q} := \mathcal{X}_\lambda^* \bar{Q}_\lambda|_{\mathcal{M}_0} = Q_0 + \lambda \mathcal{X}_\lambda^{(1)} Q + \frac{1}{2} \lambda^2 \mathcal{X}_\lambda^{(2)} Q + O(\lambda^3). \quad (1)$$

Here, ${}^{(1)}_{\mathcal{X}}Q$ (${}^{(2)}_{\mathcal{X}}Q$) are the first-order (second-order) perturbation of \bar{Q}_λ .

The diffeomorphism $\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda$ is the map $\Phi_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ and does change the point identification. So, Φ_λ is the *gauge transformation* $\Phi_\lambda : \mathcal{X}_\lambda \rightarrow \mathcal{Y}_\lambda$ and the induced pull-back operates as ${}_{\mathcal{Y}}Q_\lambda = \Phi_\lambda^* {}_{\mathcal{X}}Q_\lambda$. The generic Taylor expansion¹ leads the order-by-order gauge-transformation rules for the perturbations as

$${}^{(1)}_{\mathcal{Y}}Q - {}^{(1)}_{\mathcal{X}}Q = \mathcal{L}_{\xi_{(1)}} Q_0, \quad {}^{(2)}_{\mathcal{Y}}Q - {}^{(2)}_{\mathcal{X}}Q = 2\mathcal{L}_{\xi_{(1)}} {}^{(1)}_{\mathcal{X}}Q + \left\{ \mathcal{L}_{\xi_{(2)}} + \mathcal{L}_{\xi_{(1)}}^2 \right\} Q_0. \quad (2)$$

where $\xi_{(1)}^a$ and $\xi_{(2)}^a$ are the generators of Φ_λ .

We call the k th-order perturbation ${}^{(k)}_{\mathcal{X}}Q$ is *gauge invariant* iff ${}^{(k)}_{\mathcal{X}}Q = {}^{(k)}_{\mathcal{Y}}Q$ for any gauge choice \mathcal{X}_λ and \mathcal{Y}_λ .¹

Through these setup, we first consider the metric perturbation to construct gauge-invariant variables for higher-order perturbations.¹ The pulled-back metric $\mathcal{X}_\lambda^* \bar{g}_{ab}$ is expanded as Eq. (1): $\mathcal{X}_\lambda^* \bar{g}_{ab} = g_{ab} + \lambda \mathcal{X} h_{ab} + (\lambda^2/2) \mathcal{X}^2 h_{ab} + O^3(\lambda)$, where g_{ab} is the metric on \mathcal{M}_0 . Our starting point of the construction of gauge-invariant variables is the following assumption for h_{ab} :

Conjecture 2.1. *If there is a symmetric tensor field h_{ab} of the second rank, whose gauge transformation rule is ${}_{\mathcal{Y}}h_{ab} - {}_{\mathcal{X}}h_{ab} = \mathcal{L}_{\xi_{(1)}} g_{ab}$, then there exist a tensor field \mathcal{H}_{ab} and a vector field X^a such that h_{ab} is decomposed as $h_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab}$, where \mathcal{H}_{ab} and X^a are transformed as ${}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0$, ${}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi_{(1)}^a$ under the gauge transformation (2), respectively.*

In this conjecture, \mathcal{H}_{ab} and X^a are *gauge-invariant* and *gauge-variant* parts of the perturbation h_{ab} , respectively. If we accept Conjecture 2.1, we can recursively define gauge-invariant variables for higher-order perturbations.¹

3. An outline of a proof of Conjecture 2.1

To prove Conjecture 2.1, we assume that the background spacetimes \mathcal{M}_0 admit ADM decomposition, whose metric is given by $g_{ab} = -\alpha^2(dt)_a(dt)_b + q_{ij}(dx^i + \beta^i dt)_a(dx^j + \beta^j dt)_b$. We decompose of the components $\{h_{ti}, h_{ij}\}$ of h_{ab} as

$$h_{ti} =: D_i h_{(VL)} + h_{(V)i} - \frac{2}{\alpha} (D_i \alpha - \beta^k K_{ik}) (h_{(VL)} - \Delta^{-1} D^k \partial_t h_{(TV)_k}) - \frac{2}{\alpha} M_i{}^k h_{(TV)_k}, \quad (3)$$

$$h_{ij} =: \frac{1}{n} q_{ij} h_{(L)} + D_i h_{(TV)_j} + D_j h_{(TV)_i} - \frac{2}{n} q_{ij} D^k h_{(TV)_k} + h_{(TT)ij} + \frac{2}{\alpha} K_{ij} (h_{(VL)} - \Delta^{-1} D^k \partial_t h_{(TV)_k}) - \frac{2}{\alpha} K_{ij} \beta^k h_{(TV)_k}, \quad (4)$$

$$D^i h_{(V)i} = 0, \quad q^{ij} h_{(TT)ij} = 0 = D^i h_{(TT)ij}. \quad (5)$$

where $M_i{}^j$ is defined by $M_i{}^j := -\alpha^2 K_i{}^j + \beta^j \beta^k K_{ki} - \beta^j D_i \alpha + \alpha D_i \beta^j$. Here, K_{ij} is the extrinsic curvature and D_i is the covariant derivative associate with the metric q_{ij} on $t = \text{const}$ hypersurfaces.

Here, we assumed the existence of Green functions of the elliptic derivative operators $\Delta := D^i D_i$ and $\mathcal{F} := \Delta - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) D^i - 2D^i \left\{ \frac{1}{\alpha} (D_i \alpha - \beta^j K_{ij}) \right\}$, and the existence and the uniqueness of the solution A_i to the equation

$$\mathcal{D}_j{}^k A_k + D^m \left[\frac{2}{\alpha} \tilde{K}_{mj} \left\{ \mathcal{F}^{-1} D^k \left(\frac{2}{\alpha} M_k{}^l A_l - \partial_t A_k \right) - \beta^k A_k \right\} \right] = L_j \quad (6)$$

for given a vector field L_j . We note that the relations (3)–(5) are invertible if we accept these three assumptions. These assumptions also imply that we have ignored perturbative modes which belong to the kernel of the above derivative operators and trivial solutions to Eq. (6). We call these modes as *zero modes*. The issue on the treatments of these zero modes is called *zero-mode problem*, which is a remaining problem in our formulation.

Due to Eqs. (3)–(5), the gauge-transformation rule $\mathcal{Y}h_{ab} - \mathcal{X}h_{ab} = \mathcal{L}_{\xi(1)} g_{ab}$ leads

$$\begin{aligned} \mathcal{Y}h_{(VL)} - \mathcal{X}h_{(VL)} &= \xi_t + \Delta^{-1} D^k \partial_t \xi_k, \quad \mathcal{Y}h_{(V)i} - \mathcal{X}h_{(V)i} = \partial_t \xi_i - D_i \Delta^{-1} D^k \partial_t \xi_k, \\ \mathcal{Y}h_{(L)} - \mathcal{X}h_{(L)} &= 2D^i \xi_i, \quad \mathcal{Y}h_{(TV)l} - \mathcal{X}h_{(TV)l} = \xi_l, \quad \mathcal{Y}h_{(TT)ij} - \mathcal{X}h_{(TT)ij} = 0. \end{aligned}$$

These yield the gauge-variant part X_a in Conjecture 2.1 is given by $X_a = X_t(dt)_a + X_i(dx^i)_a$ with $X_i := h_{(TV)i}$ and $X_t := h_{(VL)} - \Delta^{-1} D^k \partial_t h_{(TV)k}$. Using the variables X_t and X_i , we can construct gauge-invariant variables for h_{ab} as

$$\begin{aligned} -2\Phi &:= h_{tt} + \frac{2}{\alpha} (\partial_t \alpha + \beta^i D_i \alpha - \beta^j \beta^i K_{ij}) X_t - 2\partial_t X_t \\ &\quad + \frac{2}{\alpha} (\beta^i \beta^k \beta^j K_{kj} - \beta^i \partial_t \alpha + \alpha q^{ij} \partial_t \beta_j + \alpha^2 D^i \alpha - \alpha \beta^k D^i \beta_k - \beta^i \beta^j D_j \alpha) X_i, \\ -2n\Psi &:= h_{(L)} - 2D^i X_i, \quad \nu_i := h_{(V)i} - \partial_t X_i + D_i \Delta^{-1} D^k \partial_t X_k, \quad \chi_{ij} := h_{(TT)ij}, \end{aligned}$$

where n is the dimension of the $t = \text{const}$ hypersurface. The representations of the original components of h_{ab} in terms of these gauge-invariant variables, X_t , and X_i yield the assertion of Conjecture 2.1. \square

4. Discussion

Due to the above proof of Conjecture 2.1, we almost completed our formulation of general-relativistic higher-order gauge-invariant perturbation theories. This indicates the possibility of the wide applications of our formulation. Although our arguments do not include zero modes and these also have their physical meaning,³ we propose Conjecture 2.1 as an conjecture.

References

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